Velocity Addition in Two-Level Space-Time

by *P. Ivanov*

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The simplest hierarchical vision of space-time is to admit the existence of two separate levels, so that the spatial and temporal coordinates of the higher level would be constructed from the lower-level space-time, possibly in combination with some dynamic features. A program like that has been successfully accomplished in quantum physics, where the lower-level configuration space is said to be purely virtual, while any observables are associated with Hermitian operators acting upon the space of distributions defined on that inner space. However, one could also consider hierarchical constructs in classical physics. In this case, higher-level quantities are to be constructed in the same manner, as parameterized averages of some functions defined on the lower-level space. This not obvious, why a physicist should be interested in theories like that; however, nothing prevents us from playing with imaginary worlds just for fun. In this self-entertainment, we do not need to develop any consistent formalism, and all we are interested in is a few simplest tricks demonstrating the possibilities and limitations of the approach.

The main concern about hierarchical space-time is related to velocity definition. In a relativistic theory, we bind space to time, but still define velocities in the old 3D way, using the spatial and temporal coordinates separately as true physical values. One could wonder if such separated space and time were definable as higher-level variables, with the "closely-coupled" relativistic space-time represented by a four-dimensional coordinate system of the lower level.

Well, let us introduce a simplest "global" frame of reference representing the dynamical variables as some functions of the four coordinates, with $x^0 = ct$:

$$X^k = X^k(x^\mu), \ T = T(x^\mu) \tag{0}$$

Then we naturally obtain

$$dT = \frac{\partial T}{\partial x^\mu} dx^\mu = \tau_\mu dx^\mu$$

$$dX^k = \frac{\partial X^k}{\partial x^\mu} dx^\mu = \xi_\mu^k dx^\mu \tag{1}$$

and introduce the "global" velocity as

$$V^k = \frac{dX^k}{dT} . \tag{2}$$

To simplify things, let us always locally remain within a static, homogeneous and isotropic Minkowski space, so that there is no difference between ordinary and covariant derivatives, and the coefficients in eq. (1) are mere constants. The quantity V is related to the traditional (coordinate) velocities by a singular equation (in the non-degenerate case, known as Möbius transformation, or homography):

$$V^k = \frac{\xi_0^k + \xi_i^k \beta^i}{\tau_0 + \tau_i \beta^i} , \tag{3}$$

with the usual notation $\beta = v/c$. Indeed, every time that $\tau_0 = -\tau_i \beta^i$, the denominator in (2) becomes zero, and V may grow to infinity. This, however, may be quite tolerable as long as we do not impose any restrictions on the values of V. The obvious asymmetry in respect to the direction of velocity is

much more annoying. Why should we have singularity for negative $\tau_i \beta^i$, while a mere change in the sign of v is to eliminate any singularities and result in a smooth dependence? On the other hand, if we expect that the global velocity V is to change its sign whenever the sign of the local velocity v changes, then, for constant coefficients in (1), we have the only possibility $\xi_0 = 0$ and $\tau_i = 0$; with an appropriate normalization, this is equivalent to $V = v$. A disappointing result, isn't it?

A physicist might simply conclude that the original program is utterly unfeasible and forget about it. However, we don't do any physics here; we are just having fun. So, let us dwell a little bit on the details of the problem, assuming that we know nothing about (or just don't care for) any general algebraic results, like the equivalence of $PSL(2,\mathbf{C})$ and $SO^+(1,3)$.

The sign of velocity will change to the opposite in two cases: either the direction of the spatial displacement changes independently of the time variable, or the direction of time axis gets reversed with spatial coordinates left intact. This transform is essentially non-covariant, just like the very definition of velocity. That is, any symmetry in respect to such a transform should be considered on the local level as non-physical. In other words, the replacement of local velocity v with its opposite $-v$ in one reference frame will result in an asymmetric change in any other, and therefore the original symmetry will necessarily be broken.

Of course, one would not much bother about velocity reversal in very complex systems exhibiting quasi-chaotic behavior, or in essentially collective modes on motion. Such systems are essentially asymmetric in time (though the question about the locality of this time remains open). However, speaking about a free particle in an empty space, one can hardly expect any spatial anisotropy, even admitting a nontrivial procedure of speed measurement. Mirror reflection in the coordinate space must be equivalent to mirror reflection of motion, as long as we accept the very idea of mapping motion to a coordinate system.

The tricky point is that eq. (3) with constant coefficients does not entirely correspond to the original eq. (1) and the symmetries of the global transform (0). Indeed, using the differential representation (1) alone in the definition of velocity (3) is only admissible for continuous displacements, while any discrete transformations must be treated separately. That is, imposing any discrete symmetries on the functions in (0) will result in *pseudo-constant* coefficients in (1) and (3), so that applying any discrete transforms to eq. (3) must be complemented with the corresponding transforms for the parameters ξ and τ. This is formally equivalent to considering an *extended* local velocity: $v \to \langle v, q \rangle$ with the signs of the "orientation parameter" q correlated with the sign of velocity in mirror reflection. Obviously, the resulting algebra will be quite different from mere homography.

But let the formal aspects to mathematicians. On the intuitive level (which is only acceptable for our fun story), we can simply demand that $X^k(x^\mu)$ change their sign under local spatial inversions, while $T(x^\mu)$ be even. From the definition of the coefficients in (1), we immediately obtain that ξ and τ_0 do not depend on spatial inversions, while ξ_0 and τ will change their signs. In this case, the "global" velocity defined by eq. (3) is obviously odd in respect to local velocity reversal.

The varying signs of the coefficients in eq. (1) can be interpreted in different ways. Thus, the local coordinates might be treated as "spinors" rather than numbers, with the components corresponding to the opposite orientations of the coordinate axes. This idea is physically attractive since it refers to the structure of measurement. Alternatively, one could observe that, for a nontrivial definition of "global" velocities, eq. (1) must be essentially *nonlinear*. Probably, the weakest nonlinearity is introduced through "oriented" coefficients. Yet another possibility is to introduce higher-order terms in

displacements, for instance, using quadratic forms in $dx^\mu dx^\nu$ rather than linear expansions. In this approach, the velocity-like zero-order combinations dx^μ / dx^ν may also add to the overall nonlinearity. So far, this seems to be the most physical version of the functional introduction of "global" velocities, like in eq. (0). However, we'll play with the quasi-linear expansions a little longer, trying to explicitly demonstrate the character of dependencies.

Just for simplicity, let us stick to one-dimensional motion, so that eq. (3) become

$$V = \frac{\xi_0 + \xi\beta}{\tau_0 + \tau\beta}. \tag{4}$$

For $\tau_0 = 0$, we obtain a simple inversion

$$V = \frac{c\xi_0 / \tau}{v} + \frac{\xi}{\tau}. \tag{5}$$

This result is physically obvious in view of the expansion (1). Indeed, the condition $\tau_0 = 0$ means that thus defined "time" T is effectively a spatial coordinate, and therefore the definition (2) must be inverted, to obtain a quantity with the meaning of velocity proper. So, let us assume that $\tau_0 \neq 0$ in the following. To control dimensional values, observe that $[\xi_0] = [\xi] = 1$ and $[\tau_0] = [\tau] = [t / x]$. Now, eq. (4) can be simplified to give

$$V = \frac{V_0 + V_x\beta}{1 + q\beta}, \quad \beta = \frac{V - V_0}{V_x - qV}, \tag{6}$$

with

$$V_0 = \frac{\xi_0}{\tau_0}, \quad V_x = \frac{\xi}{\tau_0}, \quad q = \frac{\tau}{\tau_0}. \tag{7}$$

One could be tempted to further simplify this equation in the non-degenerate case, $V_0 \neq 0$, substituting $w = q\beta$, $W = V / V_0$, and $a = V_x / qV_0$, to obtain

$$W = \frac{1 + aw}{1 + w}, \quad w = \frac{W - 1}{a - W}. \tag{8}$$

However, in this form the symmetries of the original definition become obscured, since neither w nor W do not change their sign with reverted velocity v (and hence V). For instance, the traditional relativistic velocity addition

$$\beta = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2} \tag{9}$$

must be rewritten in the w-representation as

$$w = q\frac{q_2 w_1 + q_1 w_2}{q_1 q_2 + w_1 w_2}, \tag{10}$$

to preserve the signs of the original velocities, which neither nice nor friendly. However, in the special case of $|q| = 1$ and synchronized spatial inversion over all the reference frames, $q_1 = q_2$, eq. (10) becomes just like the usual addition law (9), with the signs of w, however, never depending on frame orientation:

$$w = \frac{w_1 + w_2}{1 + w_1 w_2}. \tag{10.1}$$

Well, let us look closer at velocity addition in the local and "global" spaces. Starting from eqs. (6)

and (9), we obtain the following heavy and cumbrous "global" law:

$$V = \frac{N_0 + N_1 V_1 + N_2 V_2 + N_{12} V_1 V_2}{D_0 + D_1 V_1 + D_2 V_2 + D_{12} V_1 V_2} \tag{11}$$

with

$$
\begin{aligned}
N_0 &= V_0 (V_x^2 + V_{01} V_{02}) - V_x^2 (V_{01} + V_{02}) \\
N_1 &= V_x (V_x + q_1 V_{02}) - V_0 (V_{02} + q_1 V_x) \\
N_2 &= V_x (V_x + q_2 V_{01}) - V_0 (V_{01} + q_2 V_x) \\
N_{12} &= V_0 (1 + q_1 q_2) - V_x (q_1 + q_2) \\
D_0 &= (V_x^2 + V_{01} V_{02}) - q V_x (V_{01} + V_{02}) \\
D_1 &= q(V_x + q_1 V_{02}) - (V_{02} + q_1 V_x) \\
D_2 &= q(V_x + q_2 V_{01}) - (V_{01} + q_2 V_x) \\
D_{12} &= (1 + q_1 q_2) - q(q_1 + q_2)
\end{aligned}
\tag{12}
$$

Assuming that spatial inversion acts simultaneously in all the reference frames, we can significantly simplify this expression:

$$V = \frac{V_0 (V_0^2 - V_x^2) - (V_0^2 - V_x^2)(V_1 + V_2) + (V_0 (1 + q^2) - 2 q V_x) V_1 V_2}{(V_x^2 + V_0^2 - 2 q V_x V_0) - V_0 (1 - q^2)(V_1 + V_2) + (1 - q^2) V_1 V_2}. \tag{13}$$

Though, in general, eq. (13) is still poorly comprehensible, it can be reduced to a more transparent relation in our pet special case $|q| = 1$:

$$V = \frac{(V_x^2 - V_0^2)}{(V_x^2 + V_0^2 - 2 q V_x V_0)} (V_1 + V_2 - V_0) + \frac{2(V_0 - q V_x)}{(V_x^2 + V_0^2 - 2 q V_x V_0)} V_1 V_2, \tag{14}$$

which is further rewritten as

$$V = \frac{V_x + q V_0}{V_x - q V_0} (V_1 + V_2 - V_0) + \frac{2}{V_0 - q V_x} V_1 V_2. \tag{15}$$

Noting that spatial inversion, along with $q \rightarrow -q$, will change the signs of V_1, V_2, V_0, we observe that the sign of V will also change to the opposite, as expected.

An interesting feature of this "addition law" for "global" velocities is that it contains a term with the *product* of the original velocities, and this term cannot be removed with any choice of the transform parameters. Moreover, setting $V_x = -q V_0$, we can eliminate the term with the plain addition of velocities and get an extraordinary result:

$$V = \frac{V_1 V_2}{V_0}. \tag{16}$$

That is, relativistic velocity addition in the local space-time is equivalent to the multiplication of velocities on the "global" scale! This is where out abstract play with formulas becomes really funny.

Of course, findings like that would not at all be surprising to a mathematically minded person; but here, in out toy world, we haven't occasionally got any mathematicians, and this only adds pleasure to our ingenuous amateur diversion.

Now, let us go the other way round and assume that

$$V = V_1 + V_2 \tag{17}$$

on the "global" scale. What kind of velocity "addition" are we going to get locally? Once again, the

general case is ugly and cumbrous:

$$\beta = \frac{N_0 + N_1\beta_1 + N_2\beta_2 + N_{12}\beta_1\beta_2}{D_0 + D_1\beta_1 + D_2\beta_2 + D_{12}\beta_1\beta_2} \ , \tag{18}$$

with

$$
\begin{aligned}
N_0 &= V_{01} + V_{02} - V_0 \\
N_1 &= V_x + q_1(V_{02} - V_0) \\
N_2 &= V_x + q_2(V_{01} - V_0) \\
N_{12} &= V_x(q_1 + q_2) - V_0 q_1 q_2 \\
D_0 &= V_x - q(V_{01} + V_{02}) \\
D_1 &= V_x(q_1 - q) - q q_1 V_{02} \\
D_2 &= V_x(q_2 - q) - q q_2 V_{01} \\
D_{12} &= V_x(q_1 q_2 - q(q_1 + q_2))
\end{aligned}
\tag{19}
$$

In the above special case, we get

$$\beta = \frac{V_0 + V_x(\beta_1 + \beta_2) + q(2V_x - qV_0)\beta_1\beta_2}{(V_x - 2qV_0) - q^2 V_0(\beta_1 + \beta_2) - q^2 V_x \beta_1\beta_2} \ . \tag{20}$$

Obviously, this law is very far from anything familiar. In the trivial case of $V_0 = 0$, we obtain probably the simplest possible form:

$$\beta = \frac{\beta_1 + \beta_2 + 2q\beta_1\beta_2}{1 - q^2\beta_1\beta_2} \ . \tag{21}$$

With $|q| = 1$, one obtains an "almost relativistic" velocity addition law:

$$\beta = \frac{\beta_1 + \beta_2 + 2q\beta_1\beta_2}{1 - \beta_1\beta_2} \ . \tag{22}$$

Just like the traditional relativistic velocity addition law (9) contains a singularity at $1 + \beta_1\beta_2 = 0$, the rule (22) is singular at $1 - \beta_1\beta_2 = 0$. However, the singularity in relativistic velocity addition can be effectively removed if we admit that $|\beta| \le 1$. A similar effect is observed in (22) with $q = -1$ in the standard spatial orientation; indeed, assuming that $|\beta| \le 1$ and setting $\beta_2 = 1$, we get $\beta = 1$ regardless of β_1. To stress the presence of the upper limit, eq. (22) with $q = -1$ can be rewritten as

$$\beta = 1 - \frac{\bar{\beta}_1\bar{\beta}_2}{1 - \beta_1\beta_2} \ , \tag{23}$$

where $\bar{\beta} = 1 - \beta$. Similarly, the relativistic addition law is presented as

$$\beta = 1 - \frac{\bar{\beta}_1\bar{\beta}_2}{1 + \beta_1\beta_2} \ . \tag{24}$$

The two formulas only differ by the sign in the denominator. For smaller speeds, retaining the quadratic terms, we get

$$\beta = \beta_1 + \beta_2 - 2\beta_1\beta_2 \tag{23.1}$$

for our imaginary world, and

$$\beta = \beta_1 + \beta_2 \tag{24.1}$$

for the relativistic law. That is, the relativistic sum is faster gaining relativistic behavior, while the

assumption of simple velocity addition on the upper level generally makes lower-level velocities more classical.

For completeness, consider also the case of non-relativistic local addition, $\beta = \beta_1 + \beta_2$. On the global level, in view of eq. (6) and with equal frame orientation, this will result in

$$V = \frac{V_x^2(V_1 + V_2 - V_0) + q(qV_0 - 2V_x)V_1V_2}{V_x(V_x - 2qV_0) + q^2V_0(V_1 + V_2) - q^2V_1V_2} , \tag{25}$$

which resembles eq. (20) and is as far from anything comprehensible. Similarly, for $V_0 = 0$, we obtain a simpler expression:

$$V = \frac{V_1 + V_2 - (2q/V_x)V_1V_2}{1 - (q^2/V_x^2)V_1V_2} . \tag{26}$$

Defining $B = V/V_x$, we readily get

$$B = \frac{B_1 + B_2 - 2qB_1B_2}{1 - q^2B_1B_2} , \tag{27}$$

which is a close analog of eq. (21), with the same consequences.

To conclude, the idea of simple functional introduction of upper-level coordinates and velocities does not seem to be too productive in the physical sense, though a few qualitative observations might find some counterparts in a more rigorous theory. Physically, the upper levels of hierarchy must appear as a result of lower-level *dynamics* rather than simple kinematic relations.

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