On a Covariant Velocity Definition

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Einstein's special relativity theory is really great and certainly right—just because it describes the observable behavior of physical systems in a very accurate way. One could be unsatisfied with the interpretations and seek for a more general approach that would justify the details—but any new formalism must be compatible with the "standard" relativity in the range of its applicability.

Still, for a while, one can forget about the physical reality and play with imaginary worlds that do not correspond to anything meaningful, but are funny on themselves and can well serve as a source of harmful entertainment, without any pretense.

In this way, one could question the way we normally construct dynamical variable in relativistic mechanics, and specifically, the evaluation of velocities. Indeed, the very consideration of velocities in a relativistic theory looks queer, since it obviously violates Lorentz covariance treating the spatial coordinates differently from the temporal dimension. Yes, there is a formally introduced 4-vector of velocity

$$u^{\mu} = \frac{dx^{\mu}}{ds};$$

but it does not look like velocity proper, rather being the vector of energy-momentum scaled to the rest mass. When we speak about one observer moving relative to another, we mean a regular 3-dimensional displacement per unit time, which is a zero-order term in the geometric sense, as $dx \cdot dt^{-1}$, while such a fundamental invariant as the interval is entirely based on the second-order terms $dx^{\mu} dx^{\nu}$. That is, any function of coordinates and velocities represented by a power series of infinitesimal displacements may contain nontrivial zero-order terms, with odd powers still eliminated by the geometrical symmetries. However, in this case, why should we restrict ourselves to a single zero-order combination? Why not admit any combinations of the form dx^{μ} / dx^{ν} , which will also include terms like dt / dx , or dx_i / dx_k ? Thus, along with the traditional definition of velocity,

$$v^i = \frac{dx^i}{dt},$$

one would also consider the components of temporal density

$$\lambda_k = \frac{dt}{dx^k},$$

with the obvious constraint

$$v^i \lambda_i = 3.$$

In general, the "covariant" velocity is introduced as

$$v^{\mu}{}_{\nu} = \frac{dx^{\mu}}{dx^{\nu}} \sim \delta^{\mu}{}_{\nu} + \delta^{\mu}{}_{0} \delta_{k\nu} \frac{v^k}{c} + \delta^{0}{}_{\nu} \delta^{k\mu} \frac{c}{v^k} + \delta^{\mu}{}_{i} \delta^{k}{}_{\nu} \frac{v^i}{v^k}, \; v^{\mu}{}_{\mu} = 4.$$

A general interval preserving transform will therefore depend on v^{μ}_{ν} rather than on *v*, which would result in an absolutely covariant theory containing Lorentz covariance as a special case. In a nontrivial metric, v^{μ}_{ν} can be related to the ordinary velocity v^i in a less straightforward way, stimulating alternative generalizations of mechanics.

Since covariant velocity is obviously singular at $v = 0$, it will need an adequate physical interpretation, such as, for instance, the impossibility of the rest frame. Why not? As soon as we have admitted the intertwined space and time, any spatial relation must incorporate time, and any temporal relation must incorporate space. A particle at rest will still have some time flow, which is to effectively violate the rest state leading to spatial displacements (kind of kinematic fluctuations). That is, we can stabilize particles up to a very high precision, but never in an absolute way; there is no such thing as a perfect balance. From our everyday experience, this seems to be a quite plausible physical assumption.

Constructing the usual physical quantities (like momenta, energy *etc*.) out of the covariant velocities rather than the traditional "rest frame" velocities is still an open question. However, this technical difficulty does not seem to seriously hinder developing a sound formal scheme.

Using covariant velocities may also come useful in multidimensional-time theories, since we do no longer depend on a special choice of metric signature. The perfect symmetry between space and time can be broken in different ways, to produce very peculiar worlds as the different manifestations of the same Universe. The very existence of such a symmetry also suggests the idea of the possible transitions between the different worlds.

By the way, the values $v^{\mu}_{\ \nu}$ form a so-called reciprocal matrix (with "reciprocal" taken in a different meaning than "inverse") of the type widely used in the analytical hierarchy process in decision making technologies. Probably, this coincidence is not entirely incidental, revealing some basic regularities of the very act of measurement. The matrix of covariant velocities is consistent by construction, but certainly not positive, since all the velocity directions are equally acceptable. This circumstance may require an extension of the mathematical theory of reciprocal matrices.

In the special case, with a single spatial dimension, we have

$$v^{\mu}_{\ \nu} \to \begin{pmatrix} 1 & v/c \\ c/v & 1 \end{pmatrix},$$

with the characteristic equation $(1-\lambda)^2 = 1$, which gives $\lambda_0 = 0$, $\lambda_1 = 2$, and the corresponding eigenvectors

$$\begin{pmatrix} -v/c \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} v/c \\ 1 \end{pmatrix}.$$

This remarkable result suggests that, along with the ordinary velocity, each particle has yet another *kinematic* property, presumably related to the direction of motion in time (or a kind of chirality). In this way, for instance, the distinction of particles and anti-particles could be naturally introduced in physical theory. Also, note that these eigenvectors are not orthogonal, and their product equals $1 - v^2/c^2$. Dividing the vectors by the square root of this value, we make the overlap equal to unity. However, this singular normalization requires additional physical considerations that are not present in the original form.