NOTES ON THE MARGINS

Pavel B. Ivanov

This collection of notes was not originally intended for public. Still, why not? These selected observations might convey an atmosphere of a growing thought, and maybe wisdom. Of course, wisdom (however mathematical) is not a thing to share: everybody is to cultivate it on their own. Still, sharing the very addiction to the ways of reason is already a sound contribution of a queer philosopher into the common idea of a more human future.

* * *

Mathematics as such has nothing to do with beauty. Still, as humans are capable of discovering beauty in anything at all, they will certainly find mathematics beautiful as well.

* * *

The informal terms like *strong* and *weak* are often used in mathematics, sometimes with the opposite connotations. The most typical meaning is like in this basic topological example [П. С. Александров, *Введение в теорию множеств и общую топологию* (1977)]:

> The set of all the partial orderings on a given set X is also naturally ordered. An ordering $<_1$ is said to be stronger than an ordering $<_2$ (or, the ordering $<_2$ is weaker than the ordering $<_1$) if, for every $x, y \in X$, $x <_2 y$ implies $x <_1 y$.

Here, the meaning of *stronger* would better be rephrased as *wider*: we consider the Cartesian square of a set, picking up a range of its elements (ordered pairs) in two different ways, so that all the ordered pairs of the second ordering belong to the first, while some of the points ordered in the first sense are not connected by the second relation. Nothing to do with logic: we do not really mean any deduction; rather the inverse: the two relations are both immediately present in the same perception field. Of course, with the growth of the scope, it may be necessary to conclude about certain events in an indirect manner, using some additional regularities; still, the original definition is entirely insensitive to the way we establish the connectivity. The junction denoted by the word *implies* is, in fact, an objective relation; in a more rigorous text, one would rather say *means*.

An example of a reverse judgment in the same topological field: topological invariants are said to be “stronger” than homotopic invariants, since homomorphism follows from homeomorphism, but not the other way round.

When in comes to logic (the reflexive level), the first (*extensive*) approach can be used to refer to the *scope* of a logical function (the domain of its validity) rather than to its logical structure. Just take two statements logically connected from a layman’s perspective: *if I am a crocodile, I am not human*. Since not being human does not necessarily imply being a crocodile, the second statement is much wider in scope, and hence “stronger”. Once again, it is only the application area (the presupposed universe) that is meant. However, the first attribution is much more specific, as it provides more information, making our thought precise. Many people would call such a specification stronger, as long as we disregard the emotional load. From this (*intensive*) perspective, a weak statement accounts for just a few attributes of a thing, while s strong statement takes them all.

Both attitudes are applicable to mathematical theorems as well. For instance, the Brouwer and Schönflies extensions of the well-known Jordan theorem (every plane simple closed curve divides the plane into an interior and an exterior region) can both be considered as its strengthening, albeit incompatible with each other. On the other hand, any constructive proof is more powerful (intensive) as compared to mere proof of existence.

An obvious suggestion is to never ever speak about “weakness” and “strength”, at least in the rigorous formal discourse. A more precise wording would always be preferable. Personally, I doubt that linguistic tricks can ever lead to any practical solutions. Psychologically, proper naming may be quite suggestive. Still, there is *no* entirely formal language, and certain interpretational divergences will remain in any case. Therefore, instead of chasing shadows, we are free to use any language we like, with a sufficient (that is, not excessive) disambiguation when necessary.

* * *

Mathematics is a right tools when you’ve got a problem and don’t know a thing to start with. Just attach one nothing to another, and you’ll get at least something in the end… As soon as the work can be approached with a practiced hand, there is no need to write formulas.

* * *

Mathematical proof is a typical example of an ill-posed problem. Normally, we know the answer in advance; all what is left is to find the appropriate materials and then proceed with the development. The both aspects are far from any unicity; still, the choice of technology is of minor importance in real life, since all we need is the ready-made construction. Sometimes all the effort is in vain. This does not mean that nothing can be done; quite often, a preliminary conditioning is required, to ensure the accessibility of the goal: things may become feasible with new materials, or the operation skills just have to be improved.

* * *

S. MacLane, *Mathematics: Form and Function* (1986)

> Since a formula is just a finite well-formed sequence of symbols, there is only a denumerable number of formulas in this language. Therefore, the presentation of Peano arithmetic provide only for a denumerable number of proofs by induction! This observation has the strange result that these Peano axioms have a "non-standard" model in ZFC. For consider a language with one additional constant δ and the following denumerable list of potential additional axioms:
>
> $$0 < \delta,\ s(0) < \delta,\ ss(0) < \delta,\ sss(0) < \delta,\ ...$$
>
> Any finite subset of this list has a model. Therefore, by compactness, the whole list has a model - and in that model of the Peano axioms there is a "natural number" δ larger than any $s^n(0)$!

This is a perfect illustration of the typical two-fold delusion of most meta-mathematicians: (1) they identify a theory with the means of its presentation (language), and (2) they identify a language with its apparent form (an encoding system). In real life, a theory only *makes use* of a (no matter which) language, to express certain schemes of activity, but no language can express any activity in full; similarly, in music, we reflect a musical thought in some system of notation, but it takes a good musician to properly interpret the musical text, and such interpretations may be quite different, depending on the cultural and personal circumstances. Similarly, a sequence of characters (or sounds) has nothing to do with language on itself; it can only *represent* language under certain (social) conditions. A sequence of characters can only become language when there is a cultural association of that sequence with some activity, action or operation; by the way, this is why one language can eventually be translated into another. In other words, any finite collection of codes implies an infinity of interpretations, and there is no finite (formal) way to completely specify a particular interpretation employed in any practical context. We understand each other because we share a common culture and history; that is, because any person is virtually infinite. No physical process taken as such can convey information; to become a signal, a word must be processed by some higher-level systems able to guess the meaning behind the words.

One can also observe that any language is highly extensible, and new “words” can be easily introduced when there is no way to express something with the available vocabulary. A theory is never exhausted by finite sequences of characters from a pre-defined set. To develop a theory, we have to augment it with ever more forms of expression referring to the formal constructions to be obtained; the usage of the newly introduced constructs is then to be explained in an informal way; usually, the abstract combinations have yet to break in through collective usage, first becoming an element of a professional slang, and later, in the course of the accumulation of cultural links, being included in the common language.

This is how we virtually incorporate infinity in any theory: there is no way to formally define it via a finite formula, but as the overall idea is already culturally present we can merely invent a name for it and formally relate the new character to the rest of the system.

One could compare it with the technique of operational closure used to complete a mathematical structure (set, space, manifold, …): when a sequence of admissible operations is proved to converge to no available element, we simply take that very sequence (or, rather, a class of sequences equivalent in some sense) and treat it as an additional element of the system. This logical operation directly mimics our everyday life behavior, and hence is quite appropriate in science, provided we do not forget about the limited applicability of any abstraction at all.

* * *

Mathematics and logic are the opposites.
The may meet, but it’s not something that happens too often…

* * *

Since a real number is rather a process than a thing, what should their equality really mean? Infinitesimal differences form the classes of some equivalence, so that equal numbers belong to the same class. However, to speak of equivalence, we still need to invoke some preliminary notion of equality, in a different sense, on a different level of hierarchy. Projecting the hierarchy into the real axis, we get a logical circularity.

* * *

Euclid has once suggested 5 postulates (αἴτημα) and 9 axioms (ἀξίωμα). Is there any tractable difference behind sheer names?

Etymologically, a postulate is merely a request, a demand, a suggestion; that is, an expression of subjective preference. On the contrary, an axiom is of a certain value on itself, which invokes esteem and is assumed by default; this is also subjective (as all kinds of reflection are), yet not so movable, and never arbitrary, referring to some outer requirement. In the same, opinion is the opposite of truth: an individual predisposition *vs*. the integrity of a collective subject.

Modern mathematics fails to make the distinction. It is entirely indifferent as to the source of knowledge. Once we have proved a theorem, it may play the role of the starting point for further deduction. Beginning with different truths, we get the same: hierarchical conversion does not change the whole of the theory.

This is a valid approach as long as one can disregard any dynamics. A mathematical theory is thought of as given once and forever, in its entirety, and all its construct have already been constructed, with all the possible links fixed. All we can is to contemplate, visualize the picture, where nothing can ever depend on us. This does not boil down to mere methodological principle, but also raises ideological clashes supporting the apology of the existing economic and social system; no wonder that mathematics has always remained the queen of the sciences, serving the earthly kings hand and foot.

A somewhat more flexible attitude would keep only a portion of the foundations intact, while varying the rest to obtain different theories, of which any one can happen to be the most adequate within an appropriate development. Here, the distinction of postulates from axioms is obvious, while the provable sentences (theorems) occupy the next level of hierarchy.

The possible synthesis could restore the original uniformity in the sense of the equal admission of the different “bunches of theories” arising from a definite separation of their foundation into stable and variable parts (axioms and postulates). Thus we tend to discover complexity in the apparently simple, breaking any symmetries to return to them elsewhere and otherwise.

* * *

H. M. Hubey, *The Diagonal Infinity* (1998)

> So mathematics is also an empirical-experimental science, and these days, the development of computation engines is the perfect way for mathematicians to gain what a physicist would call intuition. [...] The formal system is a hypothesis; the theorems are predictions. Whether they are true depends on the real world.

The (analytical) computation engines (like MatLab, Maple, Scientific Workplace *etc*.), however helpful for developing mathematical intuition, can hardly link mathematics to the real world. That is why “pure” mathematicians who know little about life beyond mathematics can never be qualified enough to discuss the foundations of mathematics. The true body of a mathematical theory, its matter, is in the whole range of its applications within mathematics, in other sciences, in technology, in the arts, in philosophy, and in our everyday activities, which are the only possible criterion of truth.

For example, the argument about the difference between discrete infinity and continuum (integers *vs*. real numbers) has long since been practically resolved by the numerous cases of successful application of the formal (including mathematical) results to human activity in the real world. We do not hesitate to combine the both possibilities whenever the production environments demands such a (many-level) consideration. Thus, in quantum physics, both continuous and discrete spectra can be observed, but the description of the profiles of spectral resonances is impossible without employing the idea of discrete series of quasi-bound states interacting with a number of continua.

* * *

A silly question: what is the geometry of the real world?

It may differ, depending on what we are going to do with that world. Sometimes, one could hardly call it geometry, indeed.

* * *

Excessive accuracy is a gross blunder in physics. In mathematics, this is an everyday norm… Once invented presentation forms get uncritically applied to everything, as a standard of rigor; on the face of it, just lots of precision about a rather vague something (if anything at all).

* * *

In philosophy, quality can in no way be measured, even in terms of binary discrimination. Indeed, the presence or absence of something is already a quantity, that is, the assessment of the availability of a clearly detectable quality. Still, no quality can exist on itself, like ideas do not exist without matter. In any context, quality is *represented* by a particular thing that seems to possess that quality in a sense. That thing is then to be taken for the unit, the standard, or the gauge of quality, so that all the other things could be compared to the objective reference (in cognition or as a natural process). This external comparison conveys the idea of quantity.

On the next stage, things could be ordered by the degree of comparability, the level of presence of something, or the modes of manifestation. Each hierarchical structure like that is to provide a kind of scale. Within the scale, it makes no difference, which level is to be chosen for the base, and there is no principal difference between the corresponding units. In this way, the scale is related to its possible conversions, the measure (gauge) hierarchy.

* * *

All mathematics does is to impose certain restrictions on the common ways of action.

It is only in practical experience that we can judge whether such a restricted action is enough.

Probably, instead of the probability theory, we'd better develop some "theory of impossibility". The human activity is potentially infinite; still, at every moment, we need to reduce it to some finite actions, or operations, to be chosen on the basis of the current range of feasibility rather than from any general considerations. Formally, there are constraints imposed on the abstract universe (which may never exist in any actual way, as an object). We can only deal with what is consistent with these constraints. One could introduce an inner measure of constraint (the depth of restriction) as well as its outer scope (the relevant elements affected). The ration of the two (an analog of a rational number) will determine the overall dimensionality of the constraint. Of course, in general, we would speak of some distributions, or folgings, rather than numbers. It is the transition from one system of constraints to another that is of a major interest: this is what people do in their everyday life.

* * *

Mathematical materialism: there is no radius of curvature without curvature. However, it would boil down to a vulgar materialism if we stay satisfied with just that. In conscious activity, reflection becomes matter, and the other way round. In particular, one could fancy a science about the radius as such; the theory will find the essential features out such objects, and suggest the ways of observing these properties in any object of that very type, an abstract radius. This have nothing to do with other objects, like the radius of curvature. Jus because a radius as an object is not the same as the radius as a property. Coexistence of the general and specific forms of the same is quite common; however, the mutual transformations do not happen in any mystical way, without a serious activity changing both the general and the specific.

* * *

Modern mathematics exaggerates the importance of the proof, thus distracting the general attention from the point, up to entirely abandoning the matter of fact. In real life, we are rarely interested in the peculiars of the construction process; all we need is a workable construct usable in the practical tasks. The degree of our personal confidence and trust may be discussed elsewhere, but not within a specific science. The usual academic tradition has its origin in the antique and medieval scholastics, where the intended result referred to the possibility of dictate, imposing one's will, regardless of the actual state of affair. Such are the class roots of mathematics. In a different world, where one is rather apt to share any achievements with the others (never alien to any alternatives), formal reasoning will lose the status of proof and become mere heuristic, search for integrity.

* * *

В. А. Успенский, *Что такое аксиоматический метод* (2001):

> And if we prove our theorem on the basis of other, earlier proved, theorems, and then those other deduce from anything else, this reduction can never be infinite. So, we'd have to stop somewhere and abandon the idea of proving all our statements, just taking some of them for the axioms.

And a few pages later:

> That is, in the definition-based approach, the notion are defined through other notions, which are, in their turn, to be expressed in some other terms, and so on. Still, we cannot proceed like that without end. This means that some notions are to be taken as they are, to break the infinite reduction. Such primary notions that do not need any definition will be treated as *undefinable*, or *fundamental*.

What a queerest kind of logic! When we make things from other things, which are to be somehow produced as well, this does not mean that there can be things requiring no effort to produce (or at least introduce in the industry). Nothing is taken from nowhere. So, why not consider the very process of (re)production as primary? Let it be infinitely traceable to the past and extendable to the future. Does it hamper anything? In this activity-bound approach, there is no need in axioms or primary notions, and everything is equally fundamental. All we need is to demonstrate the inner integrity of our knowledge, with everything following from something, but never in an absolute sense, admitting that it may as well follow from something else. Thus, we know that any two points on a sphere can be connected with a smooth path; is there any reason for deifying the only "zero" point, with the rest of the sphere produced from that origin?

The attempts to reduce real life to a pre-defined set of rules (of course, natural and sacred) reflect the inner organization of the class society. The essence of the axiomatic method is to separate the legislators from the executives, opposing the laws to legality. In this picture, the academic institutions play the part of the judicial power, keeping those who are not too compliant out of science.

Life would be mush more amusing in a permanently developing world where any correctness is but conventional, and any proof is merely a game. We are free to adopt any rules, and we can change them at every moment. There is a portion of truth in any play; still, the truth is never restricted to science, it belong to the human life, which does not need to care for the beginning or end.

* * *

The idea of a space reflects the objective nature of the world as a whole; that is, our practical life involves a very modest portion of the available features, but we are sure of the presence of many others. It is this knowledge of what we do not know that distinguishes a being of reason from an unconscious animal. For us, a space is an entirety of possibilities; each particular activity only employs a part of them, which defines spaces of all kinds as the different aspects of the same. The type of a space is a hierarchy of the objects "embedded" in the space (or, rather, just possible in it). As we can construct some of these objects, they become the products of activity. All of that cannot exist but "locally", within reach. Any extrapolation to the whole space is obviously inconsistent: even restricting the notion of the space to the collection of the possible objects, we get a global entity in respect to these objects, which hence cannot be included in their number and may exhibit some traits inexpressible in that kind of language; this does not, of course, mean that such properties are utterly inexpressible, as there are other "theories" better suited for the purpose.

For instance, the traditional geometry deals with points, curves, surfaces and volumes as the distinct classes of objects, and it is only within a certain model approach that lines could be built of points, surfaces of lines *etc*. The more so for distances and lengths, areas and volumes. Moreover, given a well-definable metric we still cannot be sure to construct local coordinate systems, nothing to say about the overall arithmetization of the space.

Beyond the geometrical tradition, the common idea of a point (or any other localizable object) will lose any grounds, so that a new theory is to primarily refer to some non-localizable objects, which, however, remain local in the sense of a narrowed description adapted to a specific view of the world, and hence these restricted patterns cannot be unconditionally extrapolated to the space in general. In this case, we won't speak of points and lines, but rather about fields, flows, and the like. The habit of modelling such non-localizable objects within a localized framework (using distributions and sequencing) just sticks to one of the possibilities, in the manner of analytical geometry.

http://unism.pjwb.org
http://unism.pjwb.net
http://unism.narod.ru